MTH 166

Lecture-18

Partial Differential Equations (PDE)

Unit 4: **Partial Differential Equations**

(**Book:** Advanced Engineering Mathematics by R.K.Jain and S.R.K Iyengar,**Chapter-9**)

Topic:

Partial Differential Equations (PDE)

Learning Outcomes:

1. Formation of PDE by elimination of arbitrary constants

2. Formation of PDE by elimination of arbitrary functions

3. Classification of PDE: Hyperbolic, Parabolic and Elliptic

Partial Derivatives:

Earlier: If $\boldsymbol{u = f(x)}$, it means there is dependent variable *y* and one independent variable *x*.

So, we differentiate *y* with respect to *x* and denote it as: $\boldsymbol{\frac{du}{dx}}$

Now: If $\boldsymbol{u = f(x, y)}$, it means there is dependent variable *u* and two independent variable *x* and *y*. So, we can differentiate *u* with respect to *x* or y, denoted as: $\boldsymbol{\frac{\partial u}{\partial x}}$ or $\boldsymbol{\frac{\partial u}{\partial y}}$ respectively.

Standard Notations:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= u_x = p \\ \frac{\partial u}{\partial y} &= u_y = q \end{aligned}\right\} \text{These are called first order partial derivatives.}$$

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial x^2} = u_{xx} = r \\ &\frac{\partial^2 u}{\partial x \partial y} \; or \; \frac{\partial^2 u}{\partial x \partial y} = u_{xy} \; or \quad u_{yx} = s \\ &\frac{\partial^2 u}{\partial x^2} = u_{yy} = t \end{aligned}\right\} \text{These are called second order partial derivatives.}$$

Partial Differential Equation:

An equation of the form: $f(x, y, p, q) = 0$ is called first order partial differential equation.

An equation of the form: g$(x, y, p, q, r, s, t) = 0$ is called second order partial differential equation.

Methods of Formation of PDE:

1. By elimination of arbitrary constants

Problem 1. Form the PDE for: $\boldsymbol{u = ax + by,}$ $\quad$ ***a*** **and** ***b*** **are constants**

Solution. Given $u = ax + by \qquad (1) \qquad [u = f(x, y)]$

Differentiating (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = a(1) + b(0) = a \qquad \Rightarrow p = a$$

Differentiating (1) partially w.r.t y

$$\frac{\partial u}{\partial y} = a(0) + b(1) = b \qquad \Rightarrow q = b$$

Putting values of a and b in equation (1):

$u = px + qy$ $\quad$ **or** $\quad$ $u = \left(\frac{\partial u}{\partial x}\right)x + \left(\frac{\partial u}{\partial y}\right)y$ which is the required PDE.

Problem 2. **Form the PDE for**: $\boldsymbol{u = ax + by + a^4 + b^4}$, $\boldsymbol{a}$ **and** $\boldsymbol{b}$ **are constants**

Solution. Given $u = ax + by + a^4 + b^4$ (1) $[u = f(x, y)]$

Differentiating (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = a(1) + b(0) = a \qquad \Rightarrow p = a$$

Differentiating (1) partially w.r.t y

$$\frac{\partial u}{\partial y} = a(0) + b(1) = b \qquad \Rightarrow q = b$$

Putting values of a and b in equation (1):

$$u = px + qy + p^4 + q^4 \qquad \textbf{or} \qquad u = \left(\frac{\partial u}{\partial x}\right)x + \left(\frac{\partial u}{\partial y}\right)y + \left(\frac{\partial u}{\partial x}\right)^4 + \left(\frac{\partial u}{\partial y}\right)^4$$

which is required PDE.

Problem 3. **Form the PDE for:** $\boldsymbol{u = (x-\alpha)^2 + (y-\beta)^2}$**,** $\boldsymbol{\alpha}$ **and** $\boldsymbol{\beta}$ **are constants**

Solution. Given $u = (x-\alpha)^2 + (y-\beta)^2$ (1) $\quad$ $[u = f(x,y)]$

Differentiating (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = 2(x-\alpha) \quad \Rightarrow p = 2(x-\alpha) \quad \Rightarrow \frac{p}{2} = (x-\alpha)$$

Differentiating (1) partially w.r.t y

$$\frac{\partial u}{\partial y} = 2(y-\beta) \quad \Rightarrow q = 2(y-\beta) \quad \Rightarrow \frac{q}{2} = (y-\beta)$$

Putting values of $(x-\alpha)$ and $(y-\beta)$ in equation (1):

$$u = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 \quad \textbf{or} \quad 4u = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

which is required PDE.

Methods of Formation of PDE:

2. By elimination of arbitrary functions

Problem 1. **Form the PDE for:** $\boldsymbol{u = f(x^2 + y^2)}$

Solution. Given $u = f(x^2 + y^2)$ (1) $[u = f(x, y)]$

Differentiating (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = f'(x^2 + y^2)(2x) \Rightarrow p = f'(x^2 + y^2)(2x) \Rightarrow \frac{p}{2x} = f'(x^2 + y^2) \qquad (2)$$

Differentiating (1) partially w.r.t y

$$\frac{\partial u}{\partial y} = f'(x^2 + y^2)(2x) \Rightarrow q = f'(x^2 + y^2)(2y) \Rightarrow \frac{q}{2y} = f'(x^2 + y^2) \qquad (3)$$

Comparing (2) and (3), we get: $\frac{p}{2x} = \frac{q}{2y}$

$\Rightarrow py = qx$ **or** $\left(\frac{\partial u}{\partial x}\right) y = \left(\frac{\partial u}{\partial y}\right) x$ which is the required PDE.

<u>Problem 2.</u> Form the PDE for: $\boldsymbol{u = f\left(\frac{x}{y}\right)}$

<u>Solution.</u> Given $u = f\left(\frac{x}{y}\right)$ (1) $[u = f(x, y)]$

Differentiating (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) \quad \Rightarrow p = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) \quad \Rightarrow py = f'\left(\frac{x}{y}\right) \qquad (2)$$

Differentiating (1) partially w.r.t y

$$\frac{\partial u}{\partial y} = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \quad \Rightarrow q = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \quad \Rightarrow -q\,\frac{y^2}{x} = f'\left(\frac{x}{y}\right) \qquad (3)$$

Comparing (2) and (3), we get: $\frac{p}{2x} = \frac{q}{2y}$

$$\Rightarrow py = -q\,\frac{y^2}{x} \quad \Rightarrow px + qy = 0 \;\; \textbf{or} \;\; \left(\frac{\partial u}{\partial x}\right)x + \left(\frac{\partial u}{\partial y}\right)y = 0$$

which is the required PDE.

Problem 3. **Form the PDE for**: $\boldsymbol{u = f(ax + by)}$

Solution. Given $u = f(ax + by)$ (1) $[u = f(x, y)]$

Differentiating (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = f'(ax + by)(a) \qquad \Rightarrow \frac{p}{a} = f'(ax + by) \qquad (2)$$

Differentiating (1) partially w.r.t y

$$\frac{\partial u}{\partial y} = f'(ax + by)(b) \qquad \Rightarrow \frac{q}{b} = f'(ax + by) \qquad (3)$$

Comparing (2) and (3), we get: $\frac{p}{a} = \frac{q}{b}$

$\Rightarrow pb = qa$ **or** $\left(\frac{\partial u}{\partial x}\right) b = \left(\frac{\partial u}{\partial y}\right) a$

which is the required PDE.

Classification/Nature of Partial Differential Equations:

Let us consider a general second order homogeneous PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\,\frac{\partial^2 u}{\partial x \partial y} + \mathrm{C}\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} + Fu = 0 \quad \text{where} \quad u = f(x, y)$$

or

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \qquad (1)$$

Equation (1) will be:

1. Hyperbolic iff $(B^2 - 4AC) > 0$

2. Parabolic iff $(B^2 - 4AC) = 0$

3. Elliptic iff $(B^2 - 4AC) < 0$

Note: Coefficients of second order partial derivatives only decide nature of PDE.

Classify the following PDE as Hyperbolic, Parabolic or Elliptic:

Problem 1. $\frac{\partial^2 u}{\partial x \partial y} = 3\frac{\partial u}{\partial y}$

Solution. Given equ. is: $u_{xy} - 3u_y = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = 0, B = 1, C = 0$

Here $(B^2 - 4AC) = (1)^2 - 4(0)(0) = 1 > 0$

So, equation (1) is Hyperbolic.

Problem 2. $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

Solution. Given equ. is: $u_{xx} + 2u_{xy} + u_{yy} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = 1, B = 2, C = 1$

Here $(B^2 - 4AC) = (2)^2 - 4(1)(1) = 4 - 4 = 0$

So, equation (1) is Parabolic.

Problem 3. $\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x}$

Solution. Given equ. is: $u_{xx} + 3u_{yy} - u_x = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = 1, B = 0, C = 3$

Here $(B^2 - 4AC) = (0)^2 - 4(1)(3) = 0 - 12 < 0$

So, equation (1) is Elliptic.

Problem 4. $y\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 0$

Solution. Given equ. is: $yu_{xx} + 2xyu_{xy} + yu_{yy} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = y, B = 2x, C = y$

Here $(B^2 - 4AC) = (2x)^2 - 4(y)(y) = 4(x^2 - y^2)$

So, equation (1) is:

Hyperbolic iff $(x^2 - y^2) > 0$

Parabolic iff $(x^2 - y^2) = 0$

Elliptic iff $(x^2 - y^2) < 0$